SPRING 2025 MATH 590: QUIZ 10

Name:

1. Apply the Gram-Schmidt process to the vectors $v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ i \end{pmatrix}$, to obtain an orthogonal basis for $\mathbb{C}^2$, then use this to get an orthonormal basis for $\mathbb{C}^2$. (5 points)

Solution. Set $w_1 = v_1$. Using the G-S process,

$$w_2 = \begin{pmatrix} 2 \\ i \end{pmatrix} - \frac{-i}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{3i}{2} \end{pmatrix}.$$

$||w_1|| = \sqrt{2}, ||w_2|| = \frac{3\sqrt{2}}{2}$. Thus, $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $u_2 = \frac{2}{3\sqrt{2}} \begin{pmatrix} \frac{3}{2} \\ \frac{3i}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$.

2. Show that the matrix $B = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{pmatrix}$ is normal, with one real eigenvalue, then find an orthogonal (real) matrix Q such that $Q^{-1}BQ$ has the form $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$. (5 points)

Solution. Since B has real entries, $B^* = B^t$. We then have $BB^t = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 20 \end{pmatrix} = B^tB$. Moreover, $p_B(x) = (x-1)(x^2 - 4x + 20)$, so the real eigenvalue of A is $x = 1$, since $x^2 - 4x + 20$ has only complex roots.

$E_1 =$ null space of $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ -4 & 0 & 1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so that $u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a unit vector forming a basis for E_1. We get an orthonormal basis by taking $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Taking $Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have that Q is an orthogonal matrix and moreover, $Q^t = Q$. Thus,

$QBQ^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 4 \\ -4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & -4 & 2 \end{pmatrix}$, so we can take $a = 2$ and $b = -4$.